Coherent state transforms attached to generalized Bargmann spaces on the complex plane

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Abstract

We construct a family of coherent states transforms attached to generalized Bargmann spaces [*C.R. Acad.Sci.Paris, t.325,1997*] in the complex plane. This constitutes another way of obtaining the kernel of an isometric operator linking the space of square integrable functions on the real line with the *true-poly*-Fock spaces [*Oper.Theory. Adv.Appl.,v.117,2000*].

1 Introduction

The Bargmann transform, was originally introduced in 1961 by V. Bargmann [1] and was closely connected to the Heisenberg group. It has found many applications in quantum optics. Another interest on this transform lies in that it is a windowed Fourier transform [2] and as such it plays an important role in signal processing and harmonic analysis on phase space [3].

This transform can be defined as

$$\mathcal{B}\left[f\right](z) := \pi^{-\frac{1}{4}} \int_{\mathbb{R}} f(\xi)\, e^{-\frac{1}{2}\xi^2+\sqrt{2}\xi z-\frac{1}{2}z^2} d\xi, z \in \mathbb{C}. \tag{1.1}$$

It maps isometrically the space $L^2(\mathbb{R}, d\xi)$ of square integrable functions f on the real line onto the Fock space $\mathfrak{F}(\mathbb{C})$ of entire complex-valued functions which are $e^{-|z|^2}d\lambda-$square integrable, $d\lambda$ denotes the ordinary planar Lebesgue measure.

Note also that the Fock space $\mathfrak{F}(\mathbb{C})$ coincides with the null space

$$\mathcal{A}_0(\mathbb{C}) := \left\{\varphi \in L^2(\mathbb{C}, e^{-|z|^2}d\lambda), \widetilde{\Delta}\varphi = 0\right\} \tag{1.2}$$

of the second order differential operator [4]:

$$\widetilde{\Delta} := -\frac{\partial^2}{\partial z \partial \overline{z}} + \overline{z}\frac{\partial}{\partial \overline{z}}. \tag{1.3}$$

The latter constitutes (in suitable units and up to additive constant) a realization in $L^2(\mathbb{C}, e^{-|z|^2}d\lambda)$ of the Schrödinger operator describing the motion of a charged particle evolving in the complex plane $\mathbb{C}$ under influence of a normal uniform magnetic field. Its spectrum consists of eigenvalues of infinite multiplicity (*Landau levels*) of the form :

$$\epsilon_m = m,\ m = 0, 1, 2, ...$$

The corresponding eigenspaces

$$\mathcal{A}_m(\mathbb{C}) := \left\{ \varphi \in L^2(\mathbb{C}, e^{-|z|^2} d\lambda), \widetilde{\Delta}\varphi = \epsilon_m \varphi \right\} \tag{1.4}$$

are pairwise orthogonal in the Hilbert space $L^2(\mathbb{C}, e^{-|z|^2} d\lambda)$ which decomposes as

$$L^2(\mathbb{C}, e^{-|z|^2} d\lambda) = \bigoplus_{m \geq 0} \mathcal{A}_m(\mathbb{C}).$$

In this Note, the main objective is to construct for each Hilbert space $\mathcal{A}_m(\mathbb{C})$, $m = 0, 1, 2, ...$ a unitary transformation, $\mathcal{B}_m : L^2(\mathbb{R}) \to \mathcal{A}_m(\mathbb{C})$ in a such a way that for the first Hilbert space $\mathcal{A}_0(\mathbb{C})$, which is the Fock space, the constructed transform $\mathcal{B}_0$ coincides with the classical Bargmann transform $\mathcal{B}$. This will be achieved by adopting a coherent states analysis. Precisely, the constructed transforms are of the form

$$\mathcal{B}_m[f](z) = (-1)^m \left(2^m m! \sqrt{\pi}\right)^{-\frac{1}{2}} \int_{\mathbb{R}} f(\xi) e^{-\frac{1}{2}\xi^2 + \sqrt{2}\xi z - \frac{1}{2}z^2} H_m\left(\xi - \frac{z + \overline{z}}{2}\right) d\xi,$$

where $H_m(\xi) = (-1)^m e^{\xi^2} \left(\frac{d}{d\xi}\right)^m e^{-\xi^2}$ is the mth Hermite polynomial.

We should note that the expression of the transforms $\mathcal{B}_m$ coincides with the expression of a family of isometric operators linking the space $L^2(\mathbb{R})$ with the *true-poly-Fock* spaces introduced by N. L. Vasilevski [5] . Thereby, the present work constitutes another way to arrive at the result of theorem 2.5 in [5], by using a coherent states method exploiting tools of the L^2-spectral theory of the Schrödinger operator given in (1.3) .

In the next section, We review briefly the coherent states formalism we will be using. Section 3 deals with some needed facts on the L^2-spectral theory of the Schrödinger operator $\widetilde{\Delta}$. In section 4 we define a family of coherent state transforms attached to the generalized Bargmann spaces $\mathcal{A}_m(\mathbb{C})$.

2 Coherent states formalism

Here, we follow the generalization of the canonical coherent states according to the procedure in [6].

Let (X, μ) be a measure space and let $\mathfrak{H}^2 \subset L^2(X, \mu)$ be a closed subspace of infinite dimension. Let $\{\Phi_n\}_{n=0}^{\infty}$ be an orthogonal basis of $\mathfrak{H}^2$ satisfying, for arbitrary $x \in X$,

$$\omega(x) := \sum_{n=0}^{\infty} \rho_n^{-1} |\Phi_n(x)|^2 < +\infty,$$

where $\rho_n := \|\Phi_n\|^2_{L^2(X)}$. Define

$$\mathfrak{K}(x, y) := \sum_{n=0}^{\infty} \rho_n^{-1} \Phi_n(x) \overline{\Phi_n(y)}, \; x, y \in X.$$

Then, $\mathfrak{K}(x, y)$ is a reproducing kernel, $\mathfrak{H}^2$ is the corresponding reproducing kernel Hilbert space and $\omega(x) := \mathfrak{K}(x, x)$, $x \in X$.

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space with* $\dim \mathcal{H} = \infty$ *and* $\{\phi_n\}_{n=0}^{\infty}$ *be an orthonormal basis of $\mathcal{H}$. The coherent states labeled by points $x \in X$ are defined as the ket-vectors $\vartheta_x \equiv \mid x >\in \mathcal{H}$:*

$$\vartheta_x \equiv \mid x >:= (\omega(x))^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Phi_n(x)}{\sqrt{\rho_n}} \phi_n. \tag{2.1}$$

By definition, it is straightforward to show that $< \vartheta_x, \vartheta_x >_{\mathcal{H}} = 1$.

Definition 2.2. *The coherent state transform associated to the set of coherent states $(\vartheta_x)_{x \in X}$ is the isometric map*

$$W : \mathcal{H} \to \mathfrak{H}^2 \subset L^2(X, \mu) \tag{2.2}$$

defined for every $x \in X$ by

$$W[\phi](x) := (\omega(x))^{\frac{1}{2}} < \phi, \vartheta_x >_{\mathcal{H}} .$$

Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$< \phi, \psi >_{\mathcal{H}} = < W[\phi], W[\psi] >_{L^2(X)} = \int_X d\mu(x)\, \omega(x) < \phi, \vartheta_x >< \vartheta_x, \psi > .$$

Thereby, we have a resolution of the identity of $\mathcal{H}$ which can be expressed in Dirac's bra-ket notation as:

$$\mathbf{1}_{\mathcal{H}} = \int_X d\mu(x)\, \omega(x) \mid x >< x \mid,$$

and where $\omega(x)$ appears as a weight function.

Remark 2.1. Note that formula (2.1) can be considered as a generalization of the series expansion of the canonical coherent states

$$\vartheta_\zeta \equiv \mid \zeta >:= e^{-\frac{1}{2}|\zeta|^2} \sum_{k=0}^{+\infty} \frac{\zeta^k}{\sqrt{k!}} \phi_k, \zeta \in \mathbb{C}$$

with $\{\phi_k\}_{k=0}^{+\infty}$ being an orthonormal basis of eigenstates of the quantum harmonic oscillator. Here, the space $\mathfrak{H}^2$ is the Fock space $\mathfrak{F}(\mathbb{C})$ and $\omega(\zeta) = \pi^{-1} e^{|\zeta|^2}, \zeta \in \mathbb{C}$.

3 The generalized Fock spaces $\mathcal{A}_m(\mathbb{C})$

As the Fock space $\mathfrak{F}(\mathbb{C})$ has $K_0(z, w) := \pi^{-1} e^{z\overline{w}}$ as reproducing kernel, we have shown [4] that the Hilbert spaces $\mathcal{A}_m(\mathbb{C})$ also have explicit reproducing kernel of the form

$$K_m(z, w) := \pi^{-1} e^{\langle z, w \rangle} L_m^{(0)}\left(|z - w|^2\right), z, w \in \mathbb{C}, \tag{3.1}$$

where $L_m^{(\alpha)}(t)$ is the Laguerre polynomial defined by the Rodriguez formula as

$$L_m^{(\alpha)}(t) = \frac{1}{m!} t^{-\alpha} e^t \left(\frac{d}{dt}\right)^m \left(t^{\alpha+m} e^{-t}\right), t \in \mathbb{R}$$

In particular, if we set $\omega_m(z) := K_m(z, z)$, then $\omega_m(z) = \pi^{-1} e^{|z|^2}, z \in \mathbb{C}$.

The spaces $\mathcal{A}_m(\mathbb{C})$ have been also used to study the spectral properties of the Cauchy transform on $L^2(\mathbb{C}, e^{-|z|^2}d\lambda)$; see [7] where the authors exhibited for each fixed $m = 0,1,2,...$ an orthogonal basis denoted $\{h_{m,p}\}_{p=0}^{+\infty}$ and defined by

$$h_{m,p}(z) := \gamma_{m,p}\, {}_1F_1\left(-\min(m,p), |m-p|+1, |z|^2\right)|z|^{|m-p|} e^{-i(m-p)\arg z} \tag{3.2}$$

where

$$\gamma_{m,p} := \frac{(-1)^{\min(m,p)}(\max(m,p))!}{(|m-p|)!},$$

and ${}_1F_1$ is the confluent hypergeometric function given by [8]:

$${}_1F_1(a,b;u) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{j=0}^{+\infty}\frac{\Gamma(a+j)}{\Gamma(b+j)}\frac{u^j}{j!}, |u| < +\infty, b \neq 0,-1,-2,\ldots.$$

Here $\Gamma(a)$ is the Euler's Gamma function such that $\Gamma(j+1) = j!$ if $j = 0,1,2,....$

Note that for $a = -n$ with n being a positive integer, the hypergeometric function ${}_1F_1$ becomes a polynomial and can be expressible in term of Laguerre polynomial according to [8]:

$${}_1F_1(-n,\alpha+1;u) = \frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}L_n^{(\alpha)}(u).$$

For our purpose we shall consider the orthogonal basis of $\mathcal{A}_m(\mathbb{C})$ in the following form

$$h_{m,p}(z) = (-1)^{\min(m,p)}(\min(m,p))!\,|z|^{|m-p|}e^{-i(m-p)\arg z}L_{\min(m,p)}^{(|m-p|)}\left(|z|^2\right), z \in \mathbb{C}, \tag{3.3}$$

with the square norm in $L^2(\mathbb{C}, e^{-|z|^2}d\lambda)$ given by

$$\rho_{m,p} := \left\|h_{m,p}\right\|^2 = \pi m! p!.$$

Remark 3.1. In [7, p. 404] the elements of the orthogonal basis given in (3.2) have been also expressed as

$$h_{m,p}(z) = \sum_{j=0}^{\min(m,p)}(-1)^j\frac{m!p!}{j!(m-j)!(p-j)!}z^{m-j}\bar{z}^{p-j}. \tag{3.4}$$

We should note these complex polynomials in (3.4) were considered also by Itô [9] in the context of complex Markov process.

4 Coherent states attached to $\mathcal{A}_m(\mathbb{C})$

In this section, we shall attach to each space $\mathcal{A}_m(\mathbb{C})$ a set coherent states via series expansion according to the procedure presented in section 2. We will also give expressions of these coherent states in a closed form by using direct calculations.

Definition 4.1. *For $m = 0,1,2,....$, the coherent states associated with the space $\mathcal{A}_m(\mathbb{C})$ and labelled by points $z \in \mathbb{C}$ are defined formally according to formula* (2.1) *as*

$$\vartheta_{z,m} \equiv | z, m >:= (\omega_m(z))^{-\frac{1}{2}}\sum_{p=0}^{+\infty}\frac{h_{m,p}(z)}{\sqrt{\rho_{m,p}}}\psi_p$$

where ψ_p are elements of a total orthonormal system of $L^2(\mathbb{R}, d\xi)$ given

$$\psi_p(\xi) := \left(\sqrt{\pi}2^p p!\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\xi^2} H_p(\xi), \; p = 0, 1, 2, ..., \quad \xi \in \mathbb{R},$$

and $H_p(\xi)$ is the pth Hermite polynomial.

Proposition 4.1. *The wave functions of these coherent states are expressed as*

$$\vartheta_{z,m}(\xi) = (-1)^m \left(2^m m! \sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{1}{2}\bar{z}^2 + \sqrt{2}\xi\bar{z} - \frac{1}{2}|z|^2 - \frac{1}{2}\xi^2} H_m\left(\xi - \frac{z + \bar{z}}{2}\right), \xi \in \mathbb{R}.$$

Proof. According to Definition 4.1, we start by writing

$$\vartheta_{z,m}(\xi) = \left(\frac{1}{\pi} e^{|z|^2}\right)^{-\frac{1}{2}} \sum_{p=0}^{+\infty} \frac{h_{m,p}(z)}{\sqrt{\pi m! p!}} \psi_p(\xi).$$

Recalling the expression of $h_{m,p}(z)$ in (3.3), then these wave functions can be rewritten as

$$\vartheta_{z,m}(\xi) = \frac{e^{-\frac{1}{2}|z|^2}}{\sqrt{m!}} \sum_{p=0}^{+\infty} \frac{(-1)^{\min(m,p)}}{\sqrt{p!}} (\min(m,p))! \, |z|^{|m-p|} e^{-i(m-p)\arg z} L_{\min(m,p)}^{(|m-p|)}\left(|z|^2\right) \psi_p(\xi).$$

The integer m being fixed, we denote by $\mathfrak{S}_m(z, \xi)$ the following series:

$$\mathfrak{S}_m(z,\xi) := \sum_{p=0}^{+\infty} \frac{(-1)^{\min(m,p)}}{\sqrt{p!}} (\min(m,p))! \, |z|^{|m-p|} e^{-i(m-p)\arg z} L_{\min(m,p)}^{(|m-p|)}\left(|z|^2\right) \psi_p(\xi)$$

and we split it into two part as

$$\begin{aligned} \mathfrak{S}_m(z,\xi) &= \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^p p! \, |z|^{m-p} e^{-i(m-p)\arg z} L_p^{(m-p)}\left(|z|^2\right) \psi_p(\xi) \\ &\quad + \sum_{p=m}^{+\infty} \frac{1}{\sqrt{p!}} (-1)^m m! \, |z|^{p-m} e^{-i(m-p)\arg z} L_m^{(p-m)}\left(|z|^2\right) \psi_p(\xi) \end{aligned}$$

This can also be written as

$$\mathfrak{S}(m,z,\xi) = \mathcal{S}_{(<\infty)}(m,z,\xi) + \mathcal{S}_{(\infty)}(m,z,\xi)$$

with

$$\begin{aligned} \mathcal{S}_{(<\infty)}(m,z,\xi) &= \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^p p! \, |z|^{m-p} e^{-i(m-p)\arg z} L_p^{(m-p)}\left(|z|^2\right) \psi_p(\xi) \\ &\quad - \sum_{p=0}^{m-1} \frac{1}{\sqrt{p!}} (-1)^m m! \, |z|^{p-m} e^{-i(m-p)\arg z} L_m^{(p-m)}\left(|z|^2\right) \psi_p(\xi) \end{aligned}$$

and

$$\mathcal{S}_{(\infty)}(m,z,\xi) = \sum_{p=0}^{+\infty} \frac{1}{\sqrt{p!}} (-1)^m m! \, |z|^{p-m} e^{-i(m-p)\arg z} L_m^{(p-m)}\left(|z|^2\right) \psi_p(\xi).$$

The finite sum $\mathcal{S}_{(<\infty)}(m,z,\xi)$ reads

$$\mathcal{S}_{(<\infty)}(m,z,\xi) = \sum_{p=0}^{m-1}\left((-1)^p\sqrt{p!}\overline{z}^{m-p}L_p^{(m-p)}\left(|z|^2\right) - (-1)^m\frac{m!}{\sqrt{p!}}z^{p-m}L_m^{(p-m)}\left(|z|^2\right)\right)\psi_p(\xi)$$

Making use of the identity [10, p. 98]:

$$L_m^{(-k)}(t) = (-t)^k\frac{(m-k)!}{m!}L_{m-k}^{(k)}(t),\ 1\leq k\leq m$$

for $k=p-m$, we write the Laguerre polynomial with upper indice $p-m<0$ as

$$L_m^{(p-m)}\left(|z|^2\right) = \left(-|z|^2\right)^{m-p}\frac{p!}{m!}L_p^{(m-p)}\left(|z|^2\right),$$

and we obtain after calculation that $\mathcal{S}_{(<\infty)}(m,z,\xi)=0$.

Now, for the infinite sum $\mathcal{S}_{(\infty)}(m,z,\xi)$, we make use of the explicit expression of the Gaussian-Hermite functions

$$\psi_p(\xi) = \left(\sqrt{\pi}2^p p!\right)^{-\frac{1}{2}}e^{-\frac{1}{2}\xi^2}H_p(\xi),\ p=0,1,2,...$$

and we obtain that

$$\begin{aligned}\mathcal{S}_{(\infty)}(m,z,\xi) &= \sum_{p=0}^{+\infty}\frac{1}{\sqrt{p!}}(-1)^m m!\,|z|^{p-m}e^{-i(m-p)\arg z}L_m^{(p-m)}\left(|z|^2\right)\frac{e^{-\frac{1}{2}\xi^2}H_p(\xi)}{\left(\sqrt{\pi}2^p p!\right)^{\frac{1}{2}}}\\ &= \left(\sqrt{\pi}\right)^{-\frac{1}{2}}(-1)^m m!e^{-\frac{1}{2}\xi^2}\mathfrak{T}_{(\infty)}(m,z,\xi)\end{aligned}$$

where

$$\mathfrak{T}_{(\infty)}(m,z,\xi) := \sum_{p=0}^{+\infty}\frac{(2^p)^{-\frac{1}{2}}}{p!}z^{p-m}L_m^{(p-m)}\left(|z|^2\right)H_p(\xi)$$

Next, we make use of following addition formula involving Laguerre and Hermite polynomials [11]:

$$\begin{aligned}&\sum_{j=-n}^{+\infty}\frac{2^{-j}\beta^{\frac{j}{2}}}{(j+n)!}(a+ib)^j L_n^{(j)}\left(\frac{\beta}{2}\left(a^2+b^2\right)\right)H_{j+n}(\xi)\\ &= \frac{1}{m!}\exp\left(-\frac{\beta}{4}(a-ib)^2+\sqrt{\beta}\xi(a-ib)\right)H_n\left(\xi-\sqrt{\beta}a\right)\end{aligned}$$

for $n=m$, $j=p-n$, $\beta=2$ and $z=a+ib\in\mathbb{C}$. This gives that

$$\mathfrak{T}_{(\infty)}(m,z,\xi) = \frac{2^{-\frac{m}{2}}}{m!}e^{-\frac{1}{2}\overline{z}^2+\sqrt{2}\xi\overline{z}}H_n\left(\xi-\frac{z+\overline{z}}{2}\right)$$

Summarizing up the above calculations, we can write successively

$$\begin{aligned}\vartheta_{z,m}(\xi) &= \frac{e^{-\frac{1}{2}|z|^2}}{\sqrt{m!}}\left(\sqrt{\pi}\right)^{-\frac{1}{2}}(-1)^m m!e^{-\frac{1}{2}\xi^2}\mathfrak{T}_{(\infty)}(m,z,\xi)\\ &= \frac{e^{-\frac{1}{2}|z|^2}}{\sqrt{m!}}\left(\sqrt{\pi}\right)^{-\frac{1}{2}}(-1)^m m!e^{-\frac{1}{2}\xi^2}\left(\frac{2^{-\frac{m}{2}}}{m!}e^{-\frac{1}{2}\overline{z}^2+\sqrt{2}\xi\overline{z}}H_m\left(\xi-\frac{z+\overline{z}}{2}\right)\right)\\ &= (-1)^m\left(2^m m!\sqrt{\pi}\right)^{-\frac{1}{2}}e^{-\frac{1}{2}\overline{z}^2+\sqrt{2}\xi\overline{z}-\frac{1}{2}|z|^2-\frac{1}{2}\xi^2}H_m\left(\xi-\frac{z+\overline{z}}{2}\right).\end{aligned}$$

The proof of Proposition 4.1 is finished. ■

Finally, according to Definition 2.2, the coherent state transform associated with the coherent states $\vartheta_{z,m}$ is the unitary map:

$$\mathcal{B}_m : L^2(\mathbb{R}, d\xi) \to \mathcal{A}_m(\mathbb{C})$$

defined by

$$\mathcal{B}_m[f](z) := (\omega_m(z))^{\frac{1}{2}} \langle f, \vartheta_{z,m} \rangle_{L^2(\mathbb{R})}, f \in L^2(\mathbb{R}, d\xi), z \in \mathbb{C}$$

Explicitly,

$$\mathcal{B}_m[f](z) = (-1)^m \left(2^m m! \sqrt{\pi}\right)^{-\frac{1}{2}} \int_{\mathbb{R}} f(\xi) e^{-\frac{1}{2}\xi^2 + \sqrt{2}\xi z - \frac{1}{2}z^2} H_m\left(\xi - \frac{z + \overline{z}}{2}\right) d\xi$$

which can be called the extended Bargmann transform of index $m = 0, 1, 2, ...$

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